

# LOWER BOUNDS FOR THE COMPLEX POLYNOMIAL HARDY–LITTLEWOOD INEQUALITY

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ABSTRACT. The Hardy–Littlewood inequality for complex homogeneous polynomials asserts that given positive integers  $m \geq 2$  and  $n \geq 1$ , if  $P$  is a complex homogeneous polynomial of degree  $m$  on  $\ell_p^n$  with  $2m \leq p \leq \infty$  given by  $P(x_1, \dots, x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha$ , then there exists a constant  $C_{\mathbb{C},m,p}^{\text{pol}} \geq 1$  (which does not depend on  $n$ ) such that

$$\left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{\mathbb{C},m,p}^{\text{pol}} \|P\|,$$

with  $\|P\| := \sup_{z \in B_{\ell_p^n}} |P(z)|$ . In this short note, among other results, we provide nontrivial lower bounds for the constants  $C_{\mathbb{C},m,p}^{\text{pol}}$ . For instance we prove that, for  $m \geq 2$  and  $2m \leq p < \infty$ ,

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq 2^{\frac{m}{p}}$$

for  $m$  even, and

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq 2^{\frac{m-1}{p}}$$

for  $m$  odd. Estimates for the case  $p = \infty$  (this is the particular case of the complex polynomial Bohnenblust–Hille inequality) were recently obtained by D. Nuñez-Alarcón in 2013.

## 1. INTRODUCTION

Let  $\mathbb{K}$  denote the field of real or complex scalars. Given  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ , define  $|\alpha| := \alpha_1 + \dots + \alpha_n$  and  $\mathbf{x}^\alpha$  stands for the monomial  $x_1^{\alpha_1} \dots x_n^{\alpha_n}$  for  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{K}^n$ . The polynomial Bohnenblust–Hille inequality (see [1, 8] and the references therein) ensures that, given positive integers  $m \geq 2$  and  $n \geq 1$ , if  $P$  is a homogeneous polynomial of degree  $m$  on  $\ell_\infty^n$  given by  $P(x_1, \dots, x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha$ , then

$$\left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\text{pol}} \|P\|$$

for some constant  $B_{\mathbb{K},m}^{\text{pol}} \geq 1$  which does not depend on  $n$  (the exponent  $\frac{2m}{m+1}$  is optimal), where  $\|P\| := \sup_{z \in B_{\ell_\infty^n}} |P(z)|$ .

The search of precise estimates of the growth of the constants  $B_{\mathbb{K},m}^{\text{pol}}$  is fundamental for different applications and remains an important open problem (see [7] and the references therein). For real scalars it was shown in [9] that

$$(1.17)^m \leq B_{\mathbb{R},m}^{\text{pol}} \leq C(\varepsilon) (2 + \varepsilon)^m,$$

where  $C(\varepsilon) (2 + \varepsilon)^m$  means that given  $\varepsilon > 0$ , there is a constant  $C(\varepsilon) > 0$  such that  $B_{\mathbb{R},m}^{\text{pol}} \leq C(\varepsilon) (2 + \varepsilon)^m$  for all  $m$ . In other words, for real scalars the hypercontractivity of  $B_{\mathbb{R},m}^{\text{pol}}$  is optimal. For complex scalars the behavior of  $B_{\mathbb{K},m}^{\text{pol}}$  is still unknown. The best information we have thus far about  $B_{\mathbb{C},m}^{\text{pol}}$  are due D. Núñez-Alarcón [14] (lower bounds) and F. Bayart, D. Pellegrino and J.B. Seoane-Sepúlveda [7] (upper bounds)

$$B_{\mathbb{C},m}^{\text{pol}} \geq \begin{cases} \left( 1 + \frac{1}{2^{m-1}} \right)^{\frac{1}{4}} & \text{for } m \text{ even;} \\ \left( 1 + \frac{1}{2^{m-1}} \right)^{\frac{m-1}{4m}} & \text{for } m \text{ odd;} \end{cases}$$

$$B_{\mathbb{C},m}^{\text{pol}} \leq C(\varepsilon) (1 + \varepsilon)^m.$$

The natural extension to  $\ell_p$  spaces of the polynomial Bohnenblust–Hille inequality is called polynomial Hardy–Littlewood inequality (see [2, 13, 15] and the references therein). More precisely, given positive integers  $m \geq 2$

and  $n \geq 1$ , if  $P$  is a homogeneous polynomial of degree  $m$  on  $\ell_p^n$  with  $2m \leq p \leq \infty$  given by  $P(x_1, \dots, x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha$ , then there exists a constant  $C_{\mathbb{K},m,p}^{\text{pol}} \geq 1$  (which does not depend on  $n$ ) such that

$$\left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{\mathbb{K},m,p}^{\text{pol}} \|P\|,$$

with  $\|P\| := \sup_{z \in B_{\ell_p^n}} |P(z)|$ . Using the generalized Kahane–Salem–Zygmund inequality (see, for instance, [1]) we can verify that the exponents  $\frac{2mp}{mp+p-2m}$  are optimal for  $2m \leq p \leq \infty$ . When  $p = \infty$ , since  $\frac{2mp}{mp+p-2m} = \frac{2m}{m+1}$ , we recover the polynomial Bohnenblust–Hille inequality. In a more general point of view this kind of results can be seen as coincidence results of the theory of absolutely summing operators (see [11]).

Very recently, the authors in collaboration with P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, D. Núñez-Alarcón, J.B. Seoane-Sepúlveda and D. M. Serrano-Rodríguez (see [3]) proved that for real scalars and  $m \geq 2$ , the constants of the polynomial Hardy–Littlewood inequality has at least an hypercontractive growth. More precisely, it was proved that, for all positive integers  $m \geq 2$  and all  $2m \leq p < \infty$ ,

$$\left( \sqrt[16]{2} \right)^m \leq 2^{\frac{m^2 p + 10m - p - 6m^2 - 4}{4mp}} \leq C_{\mathbb{R},m,p}^{\text{pol}} \leq C_{\mathbb{R},m,p}^{\text{mult}} \frac{m^m}{(m!)^{\frac{mp+p-2m}{2mp}}},$$

where  $C_{\mathbb{R},m,p}^{\text{mult}}$  are the constants of the real case of the multilinear Hardy–Littlewood inequality (for estimates of these constants see [4, 5]).

In the case of complex scalars (and concerning upper bounds) similar results were proved (see [3]):

$$1 \leq C_{\mathbb{C},m,p}^{\text{pol}} \leq C_{\mathbb{C},m,p}^{\text{mult}} \frac{m^m}{(m!)^{\frac{mp+p-2m}{2mp}}}.$$

However, there are no lower bounds for  $C_{\mathbb{C},m,p}^{\text{pol}}$  that gives us nontrivial information. In this note we provide nontrivial lower bounds for the constants of the complex case of the polynomial Hardy–Littlewood inequality. More precisely we prove that, for  $m \geq 2$  and  $2m \leq p < \infty$ ,

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq 2^{\frac{m}{p}}$$

for  $m$  even, and

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq 2^{\frac{m-1}{p}}$$

for  $m$  odd. For instance,

$$\sqrt{2} \leq C_{\mathbb{C},2,4}^{\text{pol}} \leq 3.1915.$$

## 2. THE RESULT

Let  $m \geq 2$  be an even positive integer and let  $p \geq 2m$ . Consider the 2-homogeneous polynomials  $Q_2 : \ell_p^2 \rightarrow \mathbb{C}$  and  $\widetilde{Q}_2 : \ell_\infty^2 \rightarrow \mathbb{C}$  both given by  $(z_1, z_2) \mapsto z_1^2 - z_2^2 + cz_1 z_2$ . We know from [6, 9] that

$$\|\widetilde{Q}_2\| = (4 + c^2)^{\frac{1}{2}}.$$

If we follow the lines of [14] and we define the  $m$ -homogeneous polynomial  $Q_m : \ell_p^m \rightarrow \mathbb{C}$  by  $Q_m(z_1, \dots, z_m) = z_3 \dots z_m Q_2(z_1, z_2)$  we obtain

$$\|Q_m\| \leq 2^{-\frac{m-2}{p}} \|Q_2\| \leq 2^{-\frac{m-2}{p}} \|\widetilde{Q}_2\| = 2^{-\frac{m-2}{p}} (4 + c^2)^{\frac{1}{2}},$$

where we use the obvious inequality

$$\|Q_2\| \leq \|\widetilde{Q}_2\|.$$

Therefore, for  $m \geq 2$  even and  $c \in \mathbb{R}$ , from the polynomial Hardy–Littlewood inequality it follows that

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq \frac{\left( 2 + |c|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}}}{2^{-\frac{m-2}{p}} (4 + c^2)^{\frac{1}{2}}}.$$

If

$$c > \left( \frac{2^{\frac{2p+4-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1 - 2^{-\frac{2m-4}{p}}} \right)^{\frac{1}{2}},$$

it is not too difficult to prove that

$$2^{-\frac{m-2}{p}} (4 + c^2)^{\frac{1}{2}} < \left( \left( 2^{\frac{mp+p-2m}{2mp}} \right)^2 + c^2 \right)^{\frac{1}{2}},$$

i.e.,

$$2^{-\frac{m-2}{p}} (4 + c^2)^{\frac{1}{2}} < \left\| \left( 2^{\frac{mp+p-2m}{2mp}}, c \right) \right\|_2.$$

Since  $\frac{2mp}{mp+p-2m} \leq 2$ , we know that  $\ell_{\frac{2mp}{mp+p-2m}} \subset \ell_2$  and  $\|\cdot\|_2 \leq \|\cdot\|_{\frac{2mp}{mp+p-2m}}$ . Therefore, for all

$$c > \left( \frac{2^{\frac{2p+4-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1 - 2^{-\frac{2m-4}{p}}} \right)^{\frac{1}{2}},$$

we have

$$\begin{aligned} 2^{-\frac{m-2}{p}} (4 + c^2)^{\frac{1}{2}} &< \left\| \left( 2^{\frac{mp+p-2m}{2mp}}, c \right) \right\|_2 \\ &\leq \left\| \left( 2^{\frac{mp+p-2m}{2mp}}, c \right) \right\|_{\frac{2mp}{mp+p-2m}} \\ &= \left( 2 + c^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}}, \end{aligned}$$

from which we conclude that

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq \frac{\left( 2 + c^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}}}{2^{-\frac{m-2}{p}} (4 + c^2)^{\frac{1}{2}}} > 1.$$

If  $m \geq 3$  is odd, since  $\|Q_m\| \leq \|Q_{m-1}\|$ , then we have  $\|Q_m\| \leq 2^{-\frac{m-3}{p}} (4 + c^2)^{\frac{1}{2}}$  and thus we can now proceed analogously to the even case and finally conclude that for

$$c > \left( \frac{2^{\frac{2p+6-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1 - 2^{-\frac{2m-6}{p}}} \right)^{\frac{1}{2}}$$

we have

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq \frac{\left( 2 + c^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}}}{2^{-\frac{m-3}{p}} (4 + c^2)^{\frac{1}{2}}} > 1.$$

So we have:

**Theorem 2.1.** *Let  $m \geq 2$  be a positive integer and let  $p \geq 2m$ . Then, for every  $\epsilon > 0$ ,*

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq \frac{\left( 2 + \left( \left( \frac{2^{\frac{2p+4-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1 - 2^{-\frac{2m-4}{p}}} \right)^{\frac{1}{2}} + \epsilon \right)^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}}}{2^{-\frac{m-2}{p}} \left( 4 + \left( \left( \frac{2^{\frac{2p+4-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1 - 2^{-\frac{2m-4}{p}}} \right)^{\frac{1}{2}} + \epsilon \right)^2 \right)^{\frac{1}{2}}} > 1 \quad \text{if } m \text{ is even}$$

and

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq \frac{\left( 2 + \left( \left( \frac{2^{\frac{2p+6-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1 - 2^{-\frac{2m-6}{p}}} \right)^{\frac{1}{2}} + \epsilon \right)^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}}}{2^{-\frac{m-3}{p}} \left( 4 + \left( \left( \frac{2^{\frac{2p+6-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1 - 2^{-\frac{2m-6}{p}}} \right)^{\frac{1}{2}} + \epsilon \right)^2 \right)^{\frac{1}{2}}} > 1 \quad \text{if } m \text{ is odd.}$$

However we have another approach to the problem, which is surprisingly simpler than the above approach and still seems to give best (bigger) lower bounds for the constants of the polynomial Hardy–Littlewood inequality.

**Theorem 2.2.** *Let  $m \geq 2$  be a positive integer and let  $p \geq 2m$ . Then*

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq \begin{cases} 2^{\frac{m}{p}} & \text{for } m \text{ even;} \\ 2^{\frac{m-1}{p}} & \text{for } m \text{ odd;} \end{cases}$$

*Proof.* Consider  $P_2 : \ell_p^2 \rightarrow \mathbb{C}$  the 2-homogeneous polynomial given by  $\mathbf{z} \mapsto z_1 z_2$ . Observe that

$$\|P_2\| = \sup_{|z_1|^p + |z_2|^p = 1} |z_1 z_2| = \sup_{|z| \leq 1} |z| (1 - |z|^p)^{\frac{1}{p}} = 2^{-\frac{2}{p}}.$$

More generally, if  $m \geq 2$  is even and  $P_m$  is the  $m$ -homogeneous polynomial given by  $\mathbf{z} \mapsto z_1 \cdots z_m$ , then

$$\|P_m\| \leq 2^{-\frac{m}{p}}.$$

Therefore, from the polynomial Hardy–Littlewood inequality we know that

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq \frac{\left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}}}{\|P_m\|} \geq \frac{1}{2^{-\frac{m}{p}}} = 2^{\frac{m}{p}}.$$

If  $m \geq 3$  is odd, we define again the  $m$ -homogeneous polynomial  $P_m$  given by  $\mathbf{z} \mapsto z_1 \cdots z_m$  and since  $\|P_m\| \leq \|P_{m-1}\|$ , then we have  $\|P_m\| \leq 2^{-\frac{m-1}{p}}$  and thus

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq \frac{1}{2^{-\frac{m-1}{p}}} = 2^{\frac{m-1}{p}}.$$

□

### 3. COMPARING THE ESTIMATES

The estimates of Theorem 2.1 seems to become better when  $\epsilon$  grows (this seems to be a clear sign that we should avoid the terms  $z_1^2$  and  $z_2^2$  in our approach). Making  $\epsilon \rightarrow \infty$  in Theorem 2.1 we obtain

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq \begin{cases} 2^{\frac{m-2}{p}} & \text{for } m \text{ even;} \\ 2^{\frac{m-3}{p}} & \text{for } m \text{ odd,} \end{cases}$$

which are slightly worse than the estimates from Theorem 2.2.

### 4. THE CASE $m < p < 2m$

For the case  $m < p < 2m$ , there is also a version of the polynomial Hardy–Littlewood inequalities (see [12]): there exists a constant  $C_{\mathbb{K},m,p}^{\text{pol}} \geq 1$  such that, for all positive integers  $n$  and all continuous  $m$ -homogeneous polynomial  $P : \ell_p \rightarrow \mathbb{K}$  given by  $P(x_1, \dots, x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha$  we have

$$(1) \quad \left( \sum_{|\alpha|=m}^n |a_\alpha|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq C_{\mathbb{K},m,p}^{\text{pol}} \|P\|$$

and the exponent  $\frac{p}{p-m}$  is optimal. Using a polarization argument (as, for instance in [9]), but this procedure is essentially folklore, we have:

**Proposition 4.1.** *If  $P$  is a homogeneous polynomial of degree  $m$  on  $\ell_p^n$  with  $m < p < 2m$  given by  $P(x_1, \dots, x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha$ , then*

$$\left( \sum_{|\alpha|=m} |a_\alpha|^{\frac{p}{p-m}} \right)^{\frac{p-2}{p}} \leq C_{\mathbb{K},m,p}^{\text{pol}} \|P\|$$

with

$$C_{\mathbb{K},m,p}^{\text{pol}} \leq C_{\mathbb{K},m,p}^{\text{mult}} \frac{m^m}{(m!)^{\frac{p-m}{p}}},$$

where  $C_{\mathbb{K},m,p}^{\text{mult}}$  are the constants of the multilinear Hardy–Littlewood inequality.

With the same argument used in the proof of Theorem 2.1 we obtain similar estimates for the case  $m < p < 2m$ , i.e.,

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq \begin{cases} 2^{\frac{m}{p}} & \text{for } m \text{ even;} \\ 2^{\frac{m-1}{p}} & \text{for } m \text{ odd.} \end{cases}$$

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